WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST

SEMESTER ONE 2019

TEST 1: Complex numbers

Name: ______________________

Thursday 7th March

Time: 25 minutes

Total marks: $\frac{\quad}{25} + \frac{\quad}{30} = \frac{\quad}{55}$

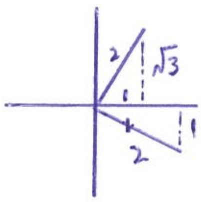
Calculator free section

1. [6 marks – 2 each]

a) Convert each of $1+\sqrt{3}\,i$ and $\sqrt{3}-i$ to polar (cis) form.

$1+\sqrt{3}\,i = 2\operatorname{cis}\frac{\pi}{3}$ ✓

$\sqrt{3}-i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$ ✓



b) Let $\omega = \dfrac{(1+\sqrt{3}\,i)^6}{(\sqrt{3}-i)^k}$. Show that $\omega = 2^{6-k}\operatorname{cis}\left(\dfrac{k\pi}{6}\right)$.

$= \operatorname{cis} 0$!

$$\omega = \frac{\left(2\operatorname{cis}\frac{\pi}{3}\right)^6}{\left(2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right)^k} = \frac{2^6\operatorname{cis}2\pi}{2^k\operatorname{cis}\left(-\frac{k\pi}{6}\right)} ✓ = 2^{6-k}\operatorname{cis}\left(0+\frac{k\pi}{6}\right) ✓$$

$$= 2^{6-k}\operatorname{cis}\frac{k\pi}{6}$$

c) For which values of k is ω purely imaginary? $(-\pi < \arg(\omega) \le \pi)$

$\frac{k\pi}{6} = \pm\frac{\pi}{2}$ ✓

$\Rightarrow k = \pm 3$ ✓

2. [7 marks – 3 and 4]

$z = a + bi$ represents a complex number, with a and b both real numbers.

a) Evaluate a and b if $2z + iz = 4 - 3i$

$2a + 2bi + ai - b = 4 - 3i$ ✓

$\therefore \; 2a - b = 4$

$2b + a = -3$ ✓

$\rightarrow \; a + 2b = -3$

$4a - 2b = 8$

$5a = 5$

$a = 1, \; b = -2$ ✓

b) Develop an equation relating a and b if $\text{Re}\left(\dfrac{\bar{z}+1}{z}\right) = 1$

$$\frac{\bar{z}+1}{z} = \frac{a - bi + 1}{a + bi} \times \frac{a - bi}{a - bi} = \frac{a^2 + a - b^2 + i(\ldots\ldots)}{a^2 + b^2}$$ ✓ ✓

$$\text{Re}\left(\frac{\bar{z}+1}{z}\right) = 1 \Rightarrow \frac{a^2 + a - b^2}{a^2 + b^2} = 1$$ ✓

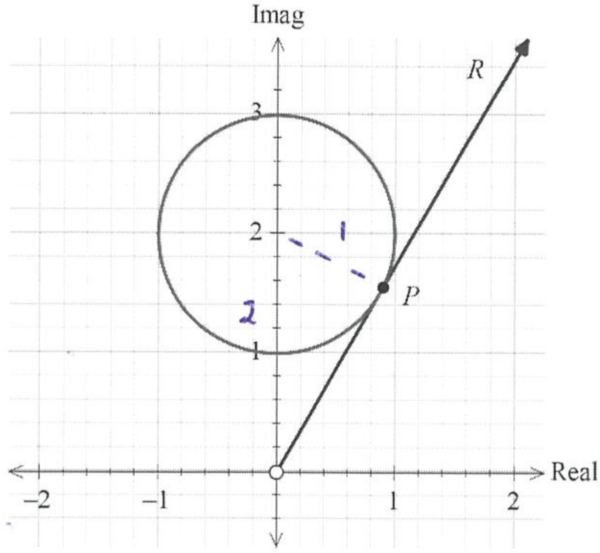
$$\Rightarrow a^2 + a - b^2 = a^2 + b^2$$

$$\Rightarrow a = 2b^2$$ ✓

3. [6 marks – 1 each]

The unit circle shown has centre $(0,2)$ and the ray R is a tangent at point P.

The circle represents a locus of complex numbers z and P is the complex number ω.

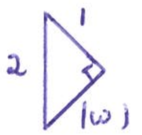


Determine:

(a) an equation for the circle, in terms of z

$|z-2i| = 1$ ✓

(b) $|\omega|$



$|\omega| = \sqrt{3}$ ✓

(c) $\arg(\omega)$

$\frac{\pi}{3}$ ✓

(d) ω expressed in Cartesian form $a+bi$

$\omega = \sqrt{3}\operatorname{cis}\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{3}{2}i$ ✓

(e) an equation for R, in the form $\text{Im}(z) = m \times \text{Re}(z) + c$, for $\text{Re}(z) > 0$

$m = \sqrt{3}$

$\Rightarrow \text{Im}(z) = \sqrt{3}\,\text{Re}(z) + 0$. ✓

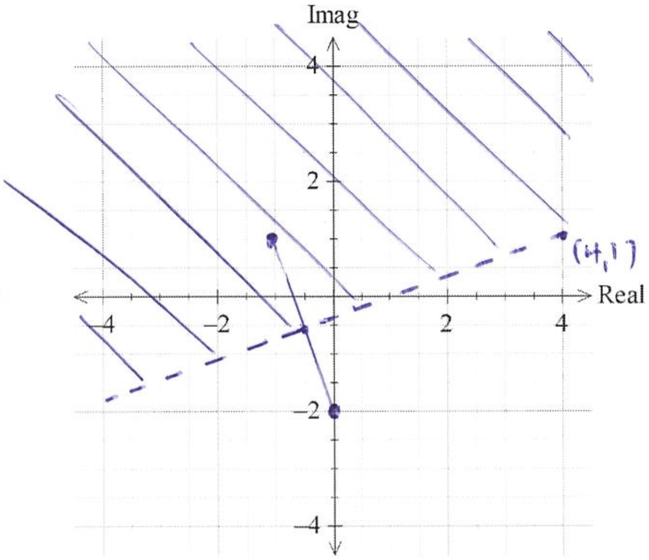
(f) the maximum value of $\arg(z)$ for the circle.

by symmetry: $\arg(z) \leq \frac{2\pi}{3}$ ✓

4. [6 marks – 3 each]

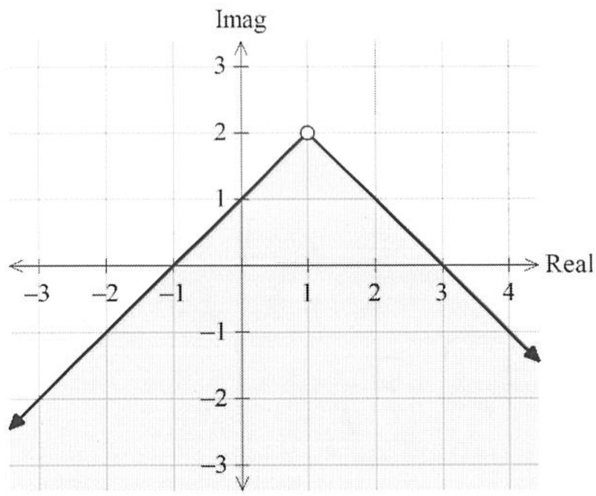
(a) Sketch, on the axes provided, the region which satisfies $\dfrac{|z+2i|}{|z+1-i|} > 1$

✓ midpt
✓ line
✓ shading



the argument of a complex number

(b) Use complex inequalities, involving $\arg(z)$, to describe the shaded region:



$$-\frac{3\pi}{4} \leq \arg(z-1-2i) \leq -\frac{\pi}{4}$$

✓ ✓ ✓

Test 1: Complex numbers

Name: ______________________

Time: 30 minutes 30 marks

Calculator assumed section

5. [9 marks – 1, 2, 3 and 3]

Let $P(z)$ be a cubic polynomial with real co-efficients. It can be written as the product of a linear factor and a quadratic factor; i.e. $P(z) = (az+b)(z^2+cz+d)$ with a, b, c and d all real.

(a) $z = 2 - i$ is a solution to $P(z) = 0$. Write down another solution.

$z_2 = 2 + i$ ✓

(b) Hence evaluate c and d.

$c = -$ sum of roots $= -4$ ✓

$d =$ product $(2+i)(2-i) = 5$ ✓

or $(z-2+i)(z-2-i) = z^2 - 4z + 5$

When $P(z)$ is divided by $(z-1)$ the remainder is 6 and when $P(z)$ is divided by $(z-2)$ the remainder is 5.

(c) Evaluate a and b.

$P(1) = 6 \Rightarrow (a+b).2 = 6$ ✓

$P(2) = 5 \Rightarrow (2a+b).1 = 5$ ✓

$a+b = 3$
$2a+b = 5$ $\Rightarrow a = 2,\ b = 1$ ✓

(d) Write $P(z)$ in expanded form (free of brackets) and hence, or otherwise, list all the zeroes of $P(z)$.

✓

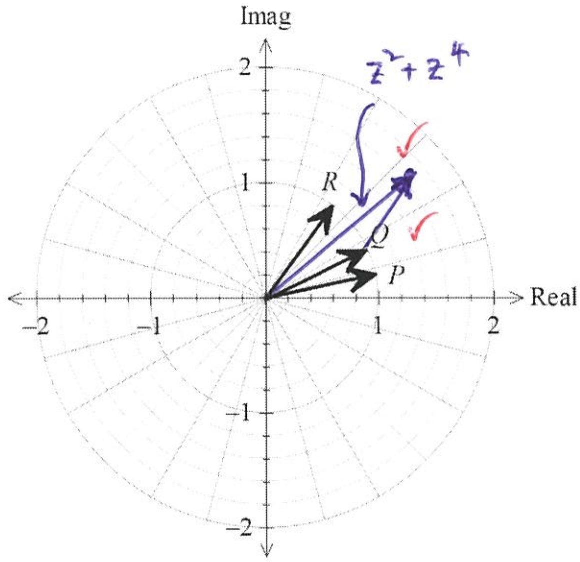
$P(z) = (2z+1)(z^2-4z+5) = 2z^3 - 7z^2 + 6z + 5$ ✓

$P(z) = 0 \Rightarrow z = 2 \pm i,\ -\frac{1}{2}$ ✓

6. [8 marks – 2, 3, 2 and 1]

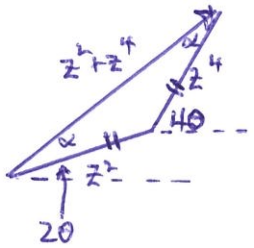
On this Argand diagram, P represents the complex number $z = \operatorname{cis}\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Q and R represent z^2 and z^4.



a) Add the complex number $z^2 + z^4$ to the diagram

(2)



b) Use the geometry of the situation, or otherwise, to show that $\arg\left(z^2+z^4\right) = 3\theta$

isosceles Δ ; 2 angles of α

3^{rd} angle $= 2\theta + \pi - 4\theta = \pi - 2\theta$ ✓

angle sum $= 2\alpha + \pi - 2\theta = \pi \Rightarrow \alpha = \theta$ ✓

$\therefore \arg(z^2+z^4) = 2\theta + \theta = 3\theta$ ✓

(or use geometry of a rhombus – diagonal bisects 2θ !)

c) Prove that $\left|z^2+z^4\right| = 2\cos\theta$

drop perp bisector of isosceles triangle ✓ $\therefore \cos\theta = \dfrac{\frac{1}{2}|z^2+z^4|}{1}$

$\Rightarrow |z^2+z^4| = 2\cos\theta$ ✓

or cosine rule : $|z^2+z^4|^2 = 1^2 + 1^2 - 2\cos(\pi - 2\theta) = 2 - 2\cos 2\theta$ ✓

$= 2 - 2(1 - 2\cos^2\theta)$

$\therefore |z^2+z^4| = \sqrt{4\cos^2\theta}$ ✓ $= 2\cos\theta$

d) Explain why $z^2 + z^4 = 2\cos\theta \operatorname{cis} 3\theta$

If $z^2+z^4 = r\operatorname{cis}\phi$ ✓

$r = |z^2+z^4| = 2\cos\theta$

$\phi = \arg(z^2+z^4) = 3\theta$

$\therefore z^2+z^4 = 2\cos\theta \operatorname{cis} 3\theta$

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7. [~~12~~ marks – 4, 1, 2, 1, 2 and 3]

a) List all the solutions to $z^5 + 1 = 0$ for $-\pi < \arg(z) \le \pi$

$z^5 = -1 = \text{cis}\,\pi$ ✓

Principal solution $z_1 = \text{cis}\frac{\pi}{5}$ ✓

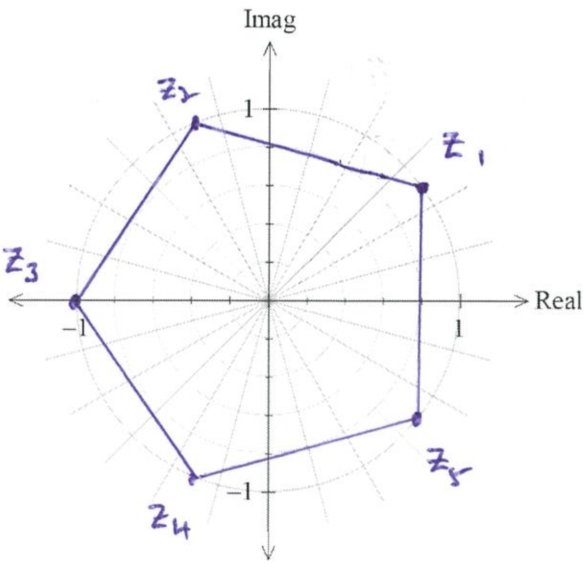
Other solutions $z_2 = \text{cis}\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) = \text{cis}\left(\frac{3\pi}{5}\right)$ ✓ 2 each.

$z_3 = \text{cis}\left(\frac{5\pi}{5}\right) = -1$ ✓

$z_4 = \text{cis}\left(\frac{\pi}{5} - \frac{2\pi}{5}\right) = \text{cis}\left(-\frac{\pi}{5}\right)$

$z_5 = \text{cis}\left(-\frac{\pi}{5} - \frac{2\pi}{5}\right) = \text{cis}\left(-\frac{3\pi}{5}\right)$

b) Represent these solutions as z_1 to z_5 on the Argand diagram, with z_1 in the first quadrant and z_5 in the fourth.



✓

c) Show that $|z_1 - z_5| = 2\sin\frac{\pi}{5}$

$|z_1 - z_5|$ = vertical distance shown ✓

$= \Delta y$

$= \sin\frac{\pi}{5} - \left(-\sin\frac{\pi}{5}\right)$ ✓

$= 2\sin\frac{\pi}{5}$

or $z_1 - z_5$

or $= \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ ✓ $- \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$= 2i\sin\frac{\pi}{5}$ ✓

$\therefore |z_1 - z_5| = 2\sin\frac{\pi}{5}$

d) Determine an expression for the perimeter of the pentagon formed by joining the solutions to $z^5 + 1 = 0$

$P = 10\sin\frac{\pi}{5}$ ✓

See over for parts e and f

e) Verify that the area of the pentagon is $\frac{5}{2}\sin\left(\frac{2\pi}{5}\right)$

$A = \frac{1}{2}ab\sin C \times 5$ ✓ for $a = b = 1$ & $\theta = \frac{2\pi}{5}$

$= \frac{5}{2}\sin\frac{2\pi}{5}$ ✓ justified

f) Generalise: determine the perimeter and area of the polygon formed by the solutions to $z^n + 1 = 0$. What happens as $n \to \infty$?

$P_n = 2n\sin\frac{\pi}{n}$ ✓

$A_n = \frac{n}{2}\sin\frac{2\pi}{n}$ ✓

as $n \to \infty$, $P_n \to 2\pi$ ✓ (circle radius 1)

$A_n \to \pi$